MATHEMATICS

CHARACTERIZATION OF A CLASS OF CUBIC FORMS

BY

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In the theory of exceptional simple Jordan algebras one encounters certain cubic forms, or what amounts to the same, symmetric trilinear forms. These forms are needed in the geometric algebra of the Moufang planes, they also are important in the study of algebraic groups of type E_6.

It is a reasonable question to ask for a simple characterization of these forms. In the present paper we shall give such a characterization. Basic is formula (1), first made explicit by FREUDENTHAL [1]. The question whether (1) characterizes the cubic form was also raised by him.

Recently, another characterization of these forms has been given by SCHAFER [4]. He characterizes the forms by means of a certain composition-property (related to formula (16)).

1. Let K be a field of characteristic $\neq 2, 3$, let V and V' be finite dimensional vector spaces over K, paired by an inner product $\langle x, x'\rangle$. Assume that we have symmetric trilinear forms $\langle x, y, z\rangle$, $\langle x', y', z'\rangle$ on $V\times V\times V$, $V'\times V'\times V'$, respectively, which are not identically zero. If $x, y \in V$, there exists by duality $x\times y \in V'$ such that

$$3\langle x, y, z\rangle = \langle x, y\times z\rangle,$$

Similarly, if $x', y' \in V'$, there exists $x'\times y' \in V$ such that

$$3\langle x', y', z'\rangle = \langle x'\times y', z'\rangle.$$

We now assume that

$$\left\{\begin{aligned} &(x\times x)\times(x\times x) = \langle x, x, x\rangle x && (x \in V),\\ &(x'\times x')\times(x'\times x') = \langle x', x', x'\rangle x' && (x' \in V').\end{aligned}\right. \tag{1}$$

Here $\langle x, x, x\rangle$ is the cubic form associated with the trilinear form $\langle x, y, z\rangle$. One encounters this situation in the theory of exceptional simple Jordan algebras. More generally, let A be a finite dimensional commutative algebra over K, with unit e, with the properties considered in [5]. This means that there is a non-degenerate quadratic form Q on A, with associated bilinear form (x, y) $(=Q(x+y)-Q(x)-Q(y))$ such that

$$\left\{\begin{aligned} &Q(x^2) = (Q(x))^2 \text{ if } (x, e) = 0,\\ &(xy, z) = (x, yz),\\ &Q(e) = \tfrac{3}{2}.\end{aligned}\right. \tag{2}$$

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In [5] these algebras are studied. They turn out to be Jordan algebras. The exceptional simple Jordan algebras fall into this category and can be singled out easily. Now define on A a product $x\times y$ by

$$x\times y=xy-\tfrac{1}{2}(x,e)y-\tfrac{1}{2}(y,e)x-\tfrac{1}{2}((x,y)-(x,e)(y,e))e, \tag{3}$$

and define a symmetric trilinear form (x, y, z) on A by

$$3(x, y, z)=(x, y\times z).$$

Putting $V=V'=A$, $\langle x, x'\rangle=(x, x')$, $\langle x, y, z\rangle=(x, y, z)$ and identifying the trilinear forms on V and V', we are in the situation described above. So we have to verify (1).

This may be done using

$$(x\times x)\times y=\tfrac{1}{2}x^2y-x(xy)+\tfrac{1}{2}(x,y)x \qquad (x, y\in A), \tag{4}$$

which is another way of writing formula (16) of [5]. (4) implies (1), taking $y=x\times x$ and observing that the subalgebra $K(x)$ of K is associative and that $x(x\times x)=\langle x, x, x\rangle e$.

Somewhat more generally, suppose that A and A' are two isomorphic algebras satisfying (2). Let ϕ be an isomorphism of A' onto A. Take $V=A$, $V'=A'$ and put

$$\langle x, x'\rangle=\lambda(x, \phi(x')),\ \langle x, y, z\rangle=\lambda(x, y, z),\ \langle x', y', z'\rangle=\lambda^2(x', y', z'). \tag{5}$$

Here (x, y) is the scalar product on A, (x, y, z) $((x', y', z'))$ is the trilinear form on $A(A')$ and λ is arbitrary $\neq 0$ in K.

It follows immediately that again (1) is valid.

We shall prove that this is the general situation. So we have the following

Theorem. *Let V and V' be vector spaces over K, paired by the inner product $\langle x, x'\rangle$, both equipped with a symmetric trilinear form such that* (1) *holds. Then we can define on V and V' algebra structures satisfying* (2), *and there is an isomorphism ϕ of V' onto V such that* (5) *holds.*

Corollary. *The cubic form $\langle x, x, x\rangle$ on V is equivalent to a scalar multiple of the generic norm (x, x, x) of a Jordan algebra A satisfying* (2).

2. So let V and V' be as before. We first derive some formal consequences of (1):

$$4(x\times x)\times(x\times y)=\langle x, x, x\rangle y+3\langle x, x, y\rangle x, \tag{6}$$

$$2(x\times x)\times(y\times y)+4(x\times y)\times(x\times y)=3\langle x, x, y\rangle y+3\langle x, y, y\rangle x, \tag{7}$$

$$4x\times(x'\times(x\times x))=\langle x, x, x\rangle x'+\langle x, x'\rangle x\times x, \tag{8}$$

of course dual formulas are also valid.

(6) and (7) follow by polarization from (1). By the dual of (6) we have

$$4((x\times x)\times(x\times x))\times(x'\times(x\times x))=\langle x\times x, x\times x, x\times x\rangle x'+$$
$$+3\langle x\times x, x\times x, x'\rangle x\times x.$$

By (1), the lefthand side is $4\langle x, x\ x\rangle x\times(x'\times(x\times x))$. On the other hand we have

$$3\langle x\times x.\ x\times x,\ x'\rangle=\langle(x\times x)\times(x\times x),\ x'\rangle=\langle x,\ x,\ x\rangle\langle x,\ x'\rangle.$$

For $x'=x\times x$ this gives $\langle x\times x,\ x\times x,\ x\times x\rangle=\langle x,\ x,\ x\rangle^2$. Substituting this we get (8) for $\langle x,\ x,\ x\rangle\neq 0$ which implies the validity of (8) for all x (this proof of (8) is given in [2]).

As a consequence of (6) we notice: if $\langle x,\ x,\ x\rangle\neq 0$ $x\times y=0$, then $y=0$.

Now take any $e\in V$ with $\lambda=\langle e,\ e,\ e\rangle\neq 0$ and put

(9) $$(x,\ y)=-6\lambda^{-1}\langle x,\ y,\ e\rangle+9\lambda^{-2}\langle x,\ e,\ e\rangle\langle y,\ e,\ e\rangle.$$

This is a symmetric bilinear form on $V\times V$, which is non-degenerate. For

(10) $$(x,\ y)=\langle x,\ -2\lambda^{-1}y\times e+3\lambda^{-2}\langle y,\ e,\ e\rangle e\times e\rangle.$$

If this is 0 for all x, then $y\times e$ is a multiple of $e\times e$. By what we observed above, this implies that y is a multiple of e. But (x, e) is not identically zero, hence y must be 0.

Put $Q(x)=\frac{1}{2}(x,\ x)$. Then

(11) $$\begin{cases} Q(e)=\frac{3}{2} \\ (x,\ e)=3\lambda^{-1}\langle x,\ e,\ e\rangle,\ \ Q(x)=-3\lambda^{-1}\langle x,\ x,\ e\rangle+\frac{1}{2}(x,\ e)^2. \end{cases}$$

Now define a product on V by

(12) $$xy=4\lambda^{-1}(x\times e)\times(y\times e)+\tfrac{1}{2}((x,\ y)-(x,\ e)(y,\ e))e.$$

Proposition 1. *The algebra V, equipped with the product* (12), *has unit element e and satisfies* (2).

Proof. We know already that $Q(e)=\frac{3}{2}$.

(a) e is unit element.

By (12) we have

$$ex=xe=4\lambda^{-1}(x\times e)\times(e\times e)-(x,\ e)e=x \text{ by (6).}$$

(b) $$(xy,\ z)=(x,\ yz).$$

From (9) we find

$$(xy,\ z)=-6\lambda^{-1}\langle xy,\ z,\ e\rangle+9\lambda^{-2}\langle xy,\ e,\ e\rangle\langle z,\ e,\ e\rangle.$$

Substituting (12) into this, we get a number of terms, some of which are symmetric in x, y, z. The other ones give a contribution

$$\tfrac{3}{2}\lambda^{-1}(x,\ y)\langle z,\ e,\ e\rangle+36\lambda^{-3}\langle(x\times e)\times(y\times e),\ e,\ e\rangle\langle z,\ e,\ e\rangle.$$

Now observe that

$$4\langle(x\times e)\times(y\times e),\ e,e\rangle=4\langle(x\times e)\times(e\times e),\ y,e\rangle=\lambda\langle x,y,e\rangle+3\langle x,e,e\rangle\langle y,e,e\rangle$$

by (6). Substituting this we get that modulo symmetric terms, (xy, z) equals $\frac{3}{2}\lambda^{-1}(x, y)\langle z, e, e\rangle + 9\lambda^{-2}\langle x, y, e\rangle\langle z, e, e\rangle$. By (9) this is also symmetric.

So (xy, z) is symmetric, which implies that $(xy, z) = (x, yz)$.

(c) $$Q(x^2) = (Q(x))^2 \text{ if } (x, e) = 0.$$

By (9) we have

$$2Q(x^2) = (x^2, x^2) = -6\lambda^{-1}\langle x^2, x^2, e\rangle + 9\lambda^{-2}\langle x^2, e, e\rangle\langle x^2, e, e\rangle.$$

Now $3\lambda^{-1}\langle x^2, e, e\rangle = (x^2, e) = 2Q(x)$ by (11) and (b). (12) gives

$$-6\lambda^{-1}\langle x^2, x^2, e\rangle = -6\lambda^{-1}\langle 4\lambda^{-1}(x\times e)\times(x\times e) + Q(x)e, 4\lambda^{-1}(x\times e)\times(x\times e) + Q(x)e, e\rangle = -32\lambda^{-3}\langle e, ((x\times e)\times(x\times e))\times((x\times e)\times(x\times e))\rangle - 16\lambda^{-2}\langle (x\times e)\times(x\times e), e, e\rangle - 6Q^2(x).$$

If $(x, e) = 3\lambda^{-1}\langle x, e, e\rangle = 0$, the first term is 0 by (1). As in the proof of part (b) we get $\langle (x\times e)\times(x\times e), e, e\rangle = \frac{3}{4}\lambda\langle x, x, e\rangle = -\frac{1}{4}\lambda^2 Q(x)$ by (9). So we find

$$-6\lambda^{-1}\langle x^2, x^2, e\rangle = -2Q^2(x) \text{ and } 2Q(x^2) = 2Q^2(x) \text{ if } (x, e) = 0.$$

This proves proposition 1.

Let as before (x, y, z) be the trilinear form on V defined, as in Nr. 1, by the algebra structure.

Proposition 2. *We have* $\langle x, y, z\rangle = \lambda(x, y, z)$.

Denote by $x \star y$ the product (3) in the algebra V. Then

$$3(x, y, z) = (x, y \star z).$$

Now by (3) and (12),

$$x \star x = 4\lambda^{-1}(x\times e)\times(x\times e) - (x, e)x,$$

which by (7) and (11) gives

$$x \star x = -2\lambda^{-1}(x\times x)\times(e\times e) - (Q(x) - \tfrac{1}{2}(x, e)^2)e.$$

So

$$3(x, x, x) = (x, x\star x) = -6\lambda^{-1}\langle x\star x, x, e\rangle + 9\lambda^{-2}\langle x\star x, e, e\rangle\langle x, e, e\rangle.$$

Calculate $\langle x\star x, x, e\rangle$ as follows:

$$\langle x\star x, x, e\rangle = -2\lambda^{-1}\langle (x\times x)\times(e\times e), x, e\rangle - (Q(x) - \tfrac{1}{2}(x, e)^2)\langle x, e, e\rangle = -2\lambda^{-1}\langle (x\times e)\times(e\times e), x, x\rangle - (Q(x) - \tfrac{1}{2}(x, e)^2)\langle x, e, e\rangle.$$

By (6) and (11) this equals $-\frac{1}{2}\langle x, x, x\rangle - \frac{1}{6}\lambda(x, e)(Q(x) - \frac{1}{2}(x, e)^2)$.

One finds similarly

$$\langle x\star x, e, e\rangle = -\tfrac{1}{3}\lambda(Q(x) - \tfrac{1}{2}(x, e)^2).$$

This leads to $(x, x, x) = \lambda^{-1}\langle x, x, x\rangle$, which implies our assertion.

We now turn to the relation between V and V'. We introduce in V' an algebra structure, using the unit element $e' = e \times e$.

Take any $a \in V$ and define a linear transformation t_a of V' into V by

$$t_a(x') = -2(a \times a) \times x' + \langle a, x' \rangle a. \tag{13}$$

Similarly, for $a' \in V'$ we can define a linear transformation $t_{a'}$ of V into V'.

We have the following relations for these linear transformations

$$t_{a \times a} t_a(x') = \langle a, a, a \rangle^2 \, x' \tag{14}$$

$$t_a(x') \times t_a(x') = t_{a \times a}(x' \times x'), \tag{15}$$

and dually.

The proofs of (14) and (15) are straightforward. (14) follows from (8), (15) from (7).

(14) and (15) imply

$$\langle t_a(x'), t_a(x'), t_a(x') \rangle = \langle a, a, a \rangle^2 \, \langle x', x', x' \rangle. \tag{16}$$

Observe that if $V = V' = A$ as in Nr. 1, we have by (4)

$$t_a(x) = 2a(ax) - a^2 x.$$

In the notation of JACOBSON [3], this means that $t_a(x) = \{axa\}$. Then (16) reduces to a well-known formula, which is the starting point of [4].

We can now prove

Proposition 3. *Assume that V and V' have the algebra structures described above. Then $\phi = \lambda^{-1} t_e$ is an isomorphism of V' onto V, with inverse $\lambda^{-1} t_{e \times e}$, and $\langle x, x' \rangle = \lambda(x, \phi(x'))$.*

Proof. It follows from (14) that $\lambda^{-1} t_e$ and $\lambda^{-1} t_{e \times e}$ are non-singular and inverses of each other.

Moreover we have $t_e(e \times e) = \lambda e$, using this and (16) we get

$$(t_e(x'), t_e(y')) = \lambda^2 (x', y'). \tag{17}$$

From (15) and (17) it follows in a straightforward manner that

$$t_e(x') \, t_e(y') = \lambda t_e(x'y').$$

That $\langle x, x' \rangle = (x, t_e(x')$ follows from (10).

The three preceding propositions imply the theorem.

3. We give here a few applications of the preceding results.

(a) Let A be an algebra satisfying (2). We use the same notations as in Nr. 1. The argument in Nr. 2 shows the following.

If $a \in A$, $(a, a, a) \neq 0$, we can introduce a new product $x_a y$ in A by

$$x_a y = 4(a, a, a)^{-1} (x \times a) \times (y \times a) + \tfrac{1}{2}((x, y) - (x, a)\,(y, a))a.$$

Defining a symmetric bilinear form by

$$(x, y)_a = -6(a, a, a)^{-1}(x, y, a) + 9(a, a, a)^{-2}(x, a, a)(y, a, a)$$

it follows that with this new product we get again an algebra structure on A satisfying (2). Call the new algebra A_a. In other words, any $a \in A$ with $(a, a, a) \neq 0$ may serve as unit element for some algebra structure on A. The trilinear form $(x, y, z)_a$ equals $(a, a, a)^{-1}(x, y, z)$.

Now assume that A is a reduced exceptional simple Jordan algebra. By theorem 5 of [5], A_a is also an exceptional simple Jordan algebra. By lemma 2 and theorem 3 of [5], A_a is reduced. Moreover the coefficient algebra of A_a is isomorphic to that of A, since this coefficient algebra is uniquely determined by the trilinear form (see theorem 1 of [6]). Applying now theorem 1 of [7], we get finally that A_a and A_b are isomorphic if and only if $(x, y)_a$ and $(x, y)_b$ are equivalent. It is easily seen that this is so if and only if the bilinear forms $(a, a, a)^{-1}(x, y, a)$ and $(b, b, b)^{-1}(x, y, b)$ are equivalent.

In particular it follows that A_a and A_b are isomorphic if and only if there exists a linear transformation t of A such that

$$\begin{cases} t(a) = b \\ (t(x), t(y), t(z)) = (a, a, a)^{-1}(b, b, b)\,(x, y, z). \end{cases}$$

Another consequence is a theorem of Witt's type for these special cubic forms.

Proposition 4. *Let $a, b \in A$, $(a, a, a)(b, b, b) \neq 0$. There exists a non-singular linear transformation t of A with $t(a) = b$, $(t(x), t(y), t(z)) = (a, a, a)^{-1}(b, b, b)\,(x, y, z)$ if and only if the bilinear forms $(a, a, a)^{-1}(x, y, a)$ and $(b, b, b)^{-1}(x, y, b)$ are equivalent.*

(b) An example of spaces V and V' satisfying our requirements is given by Freudenthal [2] (section 26).

Let K be an arbitrary field of characteristic $\neq 2, 3$ (loc. cit. K is the field of real numbers, but the arguments carry over without change). Take for V and V' the set of ordered triples $x = (X_1, X_2, X_3)$ of 3×3 matrices with elements in K.

Define

$$\langle x, x, x \rangle = \det X_1 + \det X_2 + \det X_3 - \operatorname{tr}(X_1 X_2 X_3), \tag{18}$$

and dually. Put

$$\langle x, x' \rangle = \sum_{i=1}^{3} \operatorname{tr}(X_i\, {}^t X_i').$$

Then (1) holds.

It follows that the cubic form defined by (18) is, up to a scalar factor, the generic norm of some algebra A.

Now the polynomial defined by (18) is irreducible (since already the polynomial det X in 9 variables is irreducible). Just as above in (a) we see that A is a reduced exceptional simple Jordan algebra. We assert that its coefficient algebra C is the split octave algebra over K.

By lemma 4 of [6] this will follow if we have two elements $t, u \in V$, linearly independent, such that $t\times t=u\times u=t\times u=0$.

We may take

$$t=\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},0,0\right),\quad u=\left(\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},0,0\right)$$

and use e.g. formula (26.5.1) of [2].

This leads to another description of the metasymplectic geometry M investigated in [2].

Take a fixed $e\in V$ with $(e, e, e)\neq 0$, e.g. $e=(0, 0, 1)$. Consider the $x\in V$ satisfying

$$x\neq 0,\ x\times x=0,\quad <x, e\times e>=0. \tag{19}$$

Then x, taken projectively, give the points of M. Now using the results of nr. 2, we get at once that the x satisfying (19) correspond to the $x\in A$ with $x\neq 0$, $x\times x=0$, $(x, e)=0$. Now these are just the $x\in A$ with $x\neq 0$, $x^2=0$ (see e.g. [6]). This means that the points of M are the nilpotents of the (unique) exceptional simple Jordan algebra over K with the split octave algebra as coefficient algebra (taken projectively).

We do not enter here further into the study of the metasymplectic geometry along these lines. Let us only observe that the symplecta are found from the derivations ϕ of A with $\phi^2=0$.

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